Generating random grids

The Task

Given an amount $p = p_1 \cdot p_2 \cdot \dots \cdot p_n$ of random points in an n-dimensional unit-cube ($[0..1]^n$), find a regular grid[1] with grid-dimensions of $p_1 \times \dots \times p_n$.

The Algorithm

This Algorithm is a simple greedy-algorithm recursing over the dimensionality to construct one valid solution.

Choose two dimensions i and j. Sort the points by dimension i, divide them into chunks of p_j points and sort the points inside these chunks along $x_j - x_i$.

Now you connect every k-th ($0 \leq k < p_j$) point of each chunk to generate lines along the i-th dimension. By definition these paths cannot intersect each other in an projection onto the i/j-plane (see Proof).

Let the "starting point" (i.e. the smallest in dimension i on the line) be the representative for the line.

Recurse over the other dimensions without choosing dimension i again in a similar way. In the recursive call the "1:1 point merging" will become "1:1 line-merging" (and "1:1 grid-merging") where you connect the k-th component of each line/grid.

The resulting grid will be regular because no cell will overlap with another one due to careful construction[2] and each point inside gets 2 new neighbors in each of the n calls of the function resulting in $2n$ neighbors.

Application

For our thesis we only need the cases of $n = 2$ and $n = 3$, but having a higher-dimensional solution around may come in handy in the future.

Proof

It suffices to show that given 2 points of chunk i and 1 point of chunk $i+1$ there exists no point s.t. the line-segments created by these four points intersect.

[1] i.e. no intersections of cells, each point inside the grid is connected to $2n$ points

[2] TODO: Proof the same argument as above holds for arbitrary grids using only properties of the representing point

Keep in mind that the intersection is in the i/j-projection and we are working in a unit-square.

Let a, b be the points in the ith-chunk and x, y the points in the $i+1$-th chunk. Further shall a,b and x be fixed.

We know (w.l.o.g., by construction), that

- $\{a_i, b_i\} < \{x_i, y_i\}$
- $a_j - a_i < b_j - b_i$

Assume y is placed in a way that the generated lines intersect. We have two cases to consider:

1. $x_j - x_i < y_j - y_i$

In this case the algorithm says that we have to connect x to a and y to b. As we can only choose y, this point has to have a greater distance to b than the x/a-line.

$\lambda a + (1-\lambda)x$ for $\lambda \in [0..1]$ are all points on the line. So there has to $\exists p : \|p - b\| < \|y - b\|, p \in \{\lambda a + (1-\lambda)x | \lambda \in [0..1]\}$.

For comparison we choose the squared norm. Thus we yield

$$(p_i - b_i)^2 + (p_j - b_j)^2 < (y_i - b_i)^2 + (y_j - b_j)^2$$
$$p_i^2 - 2p_i b_i + b_i^2 + p_j^2 - 2p_j b_j + b_j^2 < y_i^2 - 2b_i y_i + b_i^2 + y_j^2 - 2b_j y_j + b_j^2$$
$$p_i^2 - 2p_i b_i + p_j^2 - 2p_j b_j < y_i^2 - 2b_i y_i + y_j^2 - 2b_j y_j$$